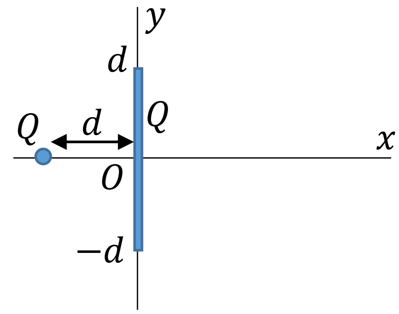
PHYS 102 – General Physics II Midterm Exam

Duration: 90 minutes Saturday, 14 October 2023; 13:30

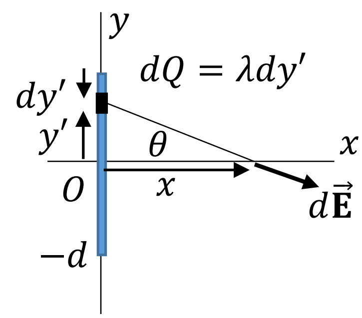
1. A uniformly charged rod of length $2d$ and total charge Q lies along the y axis between $y = -d$ and $y = d$ as shown in the figure. A point charge of magnitude Q is fixed on the x axis at $x = -d$.



(a) (20 Pts.) Find the electric field on the x axis for $x > 0$.

(b) (15 Pts.) Find the force on the rod.

Solution: (a) Linear charge density on the rod is $\lambda = Q/2d$. To find the electric field created by the line charge, we consider an infinitesimal piece dy' of the line charge at position y' with charge $dQ = \lambda dy' = Q dy'/2d$. The magnitude of the electric field created by this infinitesimal piece at a point on the x axis is



$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + y'^2} = \frac{\lambda}{4\pi\epsilon_0}\frac{dy'}{x^2 + y'^2}.$$

Because of the symmetry of the charge configuration, the electric field on the x axis has x component only, and

$$dE_x = dE\cos\theta = \frac{\lambda}{4\pi\epsilon_0}\frac{x\,dy'}{(x^2 + y'^2)^{3/2}} \quad\rightarrow\quad E_x = \frac{\lambda x}{4\pi\epsilon_0}\int_{-d}^{d}\frac{dy'}{(x^2 + y'^2)^{3/2}}.$$

The integral can be evaluated by using the substitution $y' = x\tan\theta$. The result is

$$E_x = \frac{Q}{4\pi\epsilon_0}\frac{1}{x\sqrt{x^2 + d^2}}.$$

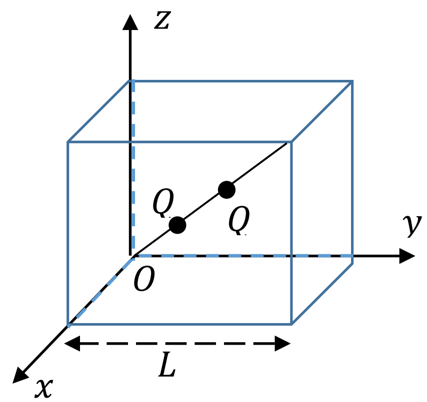
The total electric field created by the line charge and the point charge on the positive x axis is

$$\vec{\mathbf{E}}(x) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{(x + d)^2} + \frac{1}{x\sqrt{x^2 + d^2}}\right]\hat{\mathbf{i}}.$$

(b) According to Newton's third law the force on the rod by the point charge is equal in magnitude and opposite in direction to the force on the point charge by the rod. Force on the point charge Q is

$$\vec{\mathbf{F}}_Q = Q\,\vec{\mathbf{E}}_{\text{rod}}(d) = \frac{-Q^2\,\hat{\mathbf{i}}}{4\sqrt{2}\pi\epsilon_0 d^2} \quad\rightarrow\quad \vec{\mathbf{F}}_{\text{rod}} = \frac{Q^2\,\hat{\mathbf{i}}}{4\sqrt{2}\pi\epsilon_0 d^2}.$$

2. A chargeless rod of length $\sqrt{3}\,L$ has one edge at the origin and the other edge at point $x = y = z = L$. The rod forms the body diagonal of a cube with side L as shown in the figure. Two equal point charges of magnitude Q are symmetrically fixed on the rod such that the distance between the charges and the distances of the charges to the two near corners of the cube are $\sqrt{3}L/3$.



(a) (7 Pts.) What is the flux of the electric field through the cube?

(b) (7 Pts.) What is the flux of the electric field through the left side surface of the cube? (i.e., the square formed by points $(0,0,0)$, $(L,0,0)$, $(L,0,L)$, $(0,0,L)$.)

(c) (7 Pts.) What is the electric potential at the origin O?

(d) (7 Pts.) What is the electric potential at the point $(0,L,L)$?

(e) (7 Pts.) What is the potential energy of the charge configuration?

Solution: (a) Gauss's law: $\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{A} = Q_{\text{enc}}/\epsilon_0 \;\; \rightarrow \;\; \Phi_E = 2Q/\epsilon_0$.

(b) Charges are situated such that all six faces of the cube are equivalent. Therefore, flux through any face is

$$\Phi_f = \frac{1}{6}\Phi_E = \frac{Q}{3\epsilon_0}.$$

(c) Distances between the charges and the origin are $L/\sqrt{3}$ and $2L/\sqrt{3}$. Therefore, the potential at the origin is

$$V(0) = \frac{\sqrt{3}\,Q}{4\pi\epsilon_0 L} + \frac{\sqrt{3}\,Q}{4\pi\epsilon_0(2L)} \quad \rightarrow \quad V(0) = \frac{3\sqrt{3}}{2}\frac{Q}{4\pi\epsilon_0 L}.$$

(d) The unit normal in the direction of the body diagonal of the cube at the origin is $\hat{\mathbf{n}} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})/\sqrt{3}$. Therefore, position vectors of the two charges are

$$\vec{\mathbf{r}}_1 = \frac{L}{\sqrt{3}}\hat{\mathbf{n}} = \frac{L}{3}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right), \text{and } \vec{\mathbf{r}}_2 = \frac{2L}{\sqrt{3}}\hat{\mathbf{n}} = \frac{2L}{3}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right).$$

Position vector of the point $(0,L,L)$ is $\vec{\mathbf{r}} = L(\hat{\mathbf{j}} + \hat{\mathbf{k}})$. Therefore, we have

$$\vec{\mathbf{r}} - \vec{\mathbf{r}}_1 = -\frac{L}{3}\,\hat{\mathbf{i}} + \frac{2L}{3}\,\hat{\mathbf{j}} + \frac{2L}{3}\,\hat{\mathbf{k}}\,, \text{and } \vec{\mathbf{r}} - \vec{\mathbf{r}}_2 = -\frac{2L}{3}\,\hat{\mathbf{i}} + \frac{L}{3}\,\hat{\mathbf{j}} + \frac{L}{3}\,\hat{\mathbf{k}}.$$

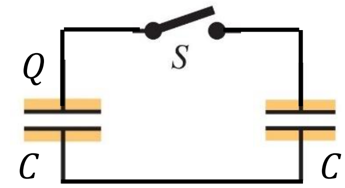
Since $|\vec{\mathbf{r}} - \vec{\mathbf{r}}_1| = L$, and $|\vec{\mathbf{r}} - \vec{\mathbf{r}}_2| = \sqrt{2}L/\sqrt{3}$, the poential at the point $(0,L,L)$ is

$$V(0,L,L) = \frac{Q}{4\pi\epsilon_0 L} + \frac{\sqrt{3}Q}{4\sqrt{2}\pi\epsilon_0 L} = \frac{(1+\sqrt{3/2})Q}{4\pi\epsilon_0 L}.$$

(e)

$$U = \frac{\sqrt{3}Q^2}{4\pi\epsilon_0 L}.$$

3. Two capacitors of equal capacitance C are connected as shown. Initially the switch S is open, and the left capacitor is charged with charge Q, while the other capacitor is uncharged. At time $t = 0$ the switch is closed.



(a) (7 Pts.) Find the charge on the left capacitor after charge equilibrium is reached.

(b) (7 Pts.) How much energy is lost when the capacitors are connected?

Now, a material with dielectric constant K is inserted to completely fill the left capacitor. The two capacitors remain connected while the material is filled.

(c) (8 Pts.) What is the final charge Q' on the left capacitor?

(d) (8 Pts.) What is the total final energy stored in both capacitors?

Solution:

(a) When the switch is closed charge flows from the charged capacitor to the uncharged capacitor until their voltages become equal. Since the capacitors are of equal capacitance $Q_f = Q/2$.

(b) İnitially $U_i = Q^2/2C$. At the final stage $U_f = Q^2/4C$. Therefore, $\Delta U = U_f - U_i = -Q^2/4C$.

(c) When a material with dielectric constant K is inserted to completely fill the left capacitor its capacitance increases to $C' = KC$, hence the voltage across it decreases. So more charge must flow to make the voltages equal. This means

$$V = \frac{Q'}{KC} = \frac{Q - Q'}{C} \quad \rightarrow \quad Q' = \frac{KQ}{1 + K}.$$

(d)

$$U_f' = \frac{Q'^2}{2KC} + \frac{(Q - Q')^2}{2C} \quad \rightarrow \quad U_f' = \frac{Q^2}{2(1 + K)C}.$$